

Super-Weyl invariance in 5D supergravity

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ABSTRACT: We propose a superspace formulation for the Weyl multiplet of $\mathcal{N} = 1$ conformal supergravity in five dimensions. The corresponding superspace constraints are invariant under super-Weyl transformations generated by a real scalar parameter. The minimal supergravity multiplet, which was introduced by Howe in 1981, emerges if one couples the Weyl multiplet to an Abelian vector multiplet and then breaks the super-Weyl invariance by imposing the gauge condition $W = 1$, with W the field strength of the vector multiplet. The geometry of superspace is shown to allow the existence of a large family of off-shell supermultiplets that possess uniquely determined super-Weyl transformation laws and can be used to describe supersymmetric matter. Many of these supermultiplets have not appeared within the superconformal tensor calculus. We formulate a manifestly locally supersymmetric and super-Weyl invariant action principle. In the super-Weyl gauge $W = 1$, this action reduces to that constructed in arXiv:0712.3102. We also present a superspace formulation for the dilaton Weyl multiplet.

KEYWORDS: Superspaces, Extended Supersymmetry, Supergravity Models.

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1. Introduction

Recently, we have constructed a superspace formulation for general $\mathcal{N} = 1$ (often called $\mathcal{N} = 2$) supergravity-matter systems [1, 2] in five space-time dimensions. In the approach of [1, 2], the geometry of curved superspace is described by the minimal supergravity multiplet introduced by Howe in 1981 [3] (see also [4]). On the other hand, if one describes 5D $\mathcal{N} = 1$ matter-coupled supergravity¹ using the component superconformal tensor calculus [8, 9], the natural starting point is the Weyl multiplet. In the latter setting, the minimal multiplet² [3] occurs by coupling the Weyl multiplet to an Abelian vector multiplet, and then breaking the Weyl invariance and some other local symmetries. To the best of our knowledge, the Weyl multiplet has never been realized in superspace.³ The present paper is aimed at filling this gap.

¹Matter couplings in 5D $\mathcal{N} = 1$ supergravity have also been studied within the on-shell component approaches [5–7].

²The minimal supergravity multiplet was re-discovered in [10] where the component implications of [3] were elaborated.

³Applying the harmonic superspace approach [11, 12] to 5D $\mathcal{N} = 1$ supergravity, it is not difficult to construct a prepotential realization for the Weyl multiplet. It is also not difficult to derive the supercurrents [4, 9] (and the multiplet of anomalies) by varying the matter action with respect to the supergravity prepotentials, similarly to the 4D $\mathcal{N} = 2$ case [13]. It is non-trivial, however, to relate the prepotential realization to an underlying covariant geometric formulation for supergravity-matter system. The latter formulation is elaborated in this paper.

Quaternion-Kähler spaces are known to be the target spaces for locally supersymmetric nonlinear sigma-models with eight supercharges [14]. As is known, there exists a one-to-one correspondence between $4n$ -dimensional quaternion-Kähler spaces and $4(n+1)$ -dimensional hyperkähler manifolds possessing a homothetic Killing vector (implying the fact that the isometry group includes a subgroup $SU(2)$ that rotates the three complex structures) [15, 16]. In the physics literature, such hyperkähler spaces are known as “hyperkähler cones” [17]. They emerge as the target spaces for rigid superconformal sigma-models with eight supercharges in diverse dimensions (see [17] and references therein). The analysis in [17] shows that in order to generate quaternion-Kähler metrics from hyperkähler cones, one essentially needs two prerequisites: (i) a superspace formulation for general rigid superconformal sigma-models with eight supercharges; (ii) a superspace extension of the superconformal tensor calculus. General rigid superconformal multiplets and their sigma-models couplings in projective superspace [18–20] have been given in [21, 22] in five and four space-time dimensions. The present paper provides the desired superspace extension of the superconformal tensor calculus in the case of five dimensions. The case of 4D $\mathcal{N} = 2$ supergravity will be considered elsewhere [23].

This paper is organized as follows. In section 2 we derive a superspace formulation for the standard Weyl multiplet in which the super-Weyl transformations are generated by an unconstrained real parameter. Section 3 is devoted to an alternative formulation in which the super-Weyl transformations are generated by a constrained real parameter. We also provide a superspace realization for the dilaton Weyl multiplet [8, 9] that corresponds to the Nishino-Rajpoot version [24] of 5D $\mathcal{N} = 1$ Poincaré supergravity. In section 4 we introduce a large family of off-shell supermultiplets that possess uniquely determined super-Weyl transformation laws and can be used to describe supersymmetric matter. Finally, in section 5 we present a manifestly locally supersymmetric and super-Weyl invariant action principle.

2. The Weyl multiplet in superspace

Let $z^{\hat{M}} = (x^{\hat{m}}, \theta^{\hat{\mu}}_i)$ be local bosonic (x) and fermionic (θ) coordinates parametrizing a curved five-dimensional $\mathcal{N} = 1$ superspace $\mathcal{M}^{5|8}$, where $\hat{m} = 0, 1, \dots, 4$, $\hat{\mu} = 1, \dots, 4$, and $i = \underline{1}, \underline{2}$. The Grassmann variables $\theta^{\hat{\mu}}_i$ are assumed to obey the standard pseudo-Majorana reality condition $(\theta^{\hat{\mu}}_i)^* = \theta^i_{\hat{\mu}} = \varepsilon_{\hat{\mu}\nu} \varepsilon^{ij} \theta^{\nu}_j$ (see the appendix of [2] for our 5D notation and conventions). The tangent-space group is chosen to be $SO(4, 1) \times SU(2)$, and the superspace covariant derivatives $\mathcal{D}_{\hat{A}} = (\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{\alpha}}^i)$ have the form

$$\mathcal{D}_{\hat{A}} = E_{\hat{A}} + \Omega_{\hat{A}} + \Phi_{\hat{A}} . \tag{2.1}$$

Here $E_{\hat{A}} = E_{\hat{A}}^{\hat{M}}(z) \partial_{\hat{M}}$ is the supervielbein, with $\partial_{\hat{M}} = \partial/\partial z^{\hat{M}}$,

$$\Omega_{\hat{A}} = \frac{1}{2} \Omega_{\hat{A}}^{\hat{b}\hat{c}} M_{\hat{b}\hat{c}} = \Omega_{\hat{A}}^{\hat{\beta}\hat{\gamma}} M_{\hat{\beta}\hat{\gamma}} , \quad M_{\hat{a}\hat{b}} = -M_{\hat{b}\hat{a}} , \quad M_{\hat{\alpha}\hat{\beta}} = M_{\hat{\beta}\hat{\alpha}} \tag{2.2}$$

is the Lorentz connection,

$$\Phi_{\hat{A}} = \Phi_{\hat{A}}^{kl} J_{kl} , \quad J_{kl} = J_{lk} \tag{2.3}$$

is the SU(2)-connection. The Lorentz generators with vector indices ($M_{\hat{a}\hat{b}}$) and spinor indices ($M_{\hat{\alpha}\hat{\beta}}$) are related to each other by the rule: $M_{\hat{a}\hat{b}} = (\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}} M_{\hat{\alpha}\hat{\beta}}$ (for more details, see the appendix of [2]). The generators of $\text{SO}(4, 1) \times \text{SU}(2)$ act on the covariant derivatives as follows:⁴

$$[J^{kl}, \mathcal{D}_{\hat{\alpha}}^i] = \varepsilon^{i(k} \mathcal{D}_{\hat{\alpha}}^{l)}, \quad [M_{\hat{\alpha}\hat{\beta}}, \mathcal{D}_{\hat{\gamma}}^k] = \varepsilon_{\hat{\gamma}(\hat{\alpha}} \mathcal{D}_{\hat{\beta}}^k, \quad [M_{\hat{a}\hat{b}}, \mathcal{D}_{\hat{c}}] = 2\eta_{\hat{c}[\hat{a}} \mathcal{D}_{\hat{b}]}, \quad (2.4)$$

where $J^{kl} = \varepsilon^{ki} \varepsilon^{lj} J_{ij}$.

The supergravity gauge group is generated by local transformations of the form

$$\delta_K \mathcal{D}_{\hat{A}} = [K, \mathcal{D}_{\hat{A}}], \quad K = K^{\hat{C}}(z) \mathcal{D}_{\hat{C}} + \frac{1}{2} K^{\hat{c}\hat{d}}(z) M_{\hat{c}\hat{d}} + K^{kl}(z) J_{kl}, \quad (2.5)$$

with all the gauge parameters obeying natural reality conditions, and otherwise arbitrary. Given a tensor superfield $\Upsilon(z)$, with its indices suppressed, it transforms as follows:

$$\delta_K U = K U. \quad (2.6)$$

The covariant derivatives obey (anti)commutation relations of the general form

$$[\mathcal{D}_{\hat{A}}, \mathcal{D}_{\hat{B}}] = T_{\hat{A}\hat{B}}^{\hat{C}} \mathcal{D}_{\hat{C}} + \frac{1}{2} R_{\hat{A}\hat{B}}^{\hat{c}\hat{d}} M_{\hat{c}\hat{d}} + R_{\hat{A}\hat{B}}^{kl} J_{kl}, \quad (2.7)$$

where $T_{\hat{A}\hat{B}}^{\hat{C}}$ is the torsion, $R_{\hat{A}\hat{B}}^{kl}$ and $R_{\hat{A}\hat{B}}^{\hat{c}\hat{d}}$ the SU(2)- and SO(4,1)-curvature tensors, respectively.

2.1 Constrained superspace geometry

We choose the torsion to obey the constraints:

$$T_{\hat{\alpha}\hat{\beta}}^{i\hat{c}} = -2i\varepsilon^{ij} (\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}} \quad (\text{dim } 0) \quad (2.8a)$$

$$T_{\hat{\alpha}\hat{\beta}\hat{k}}^{i\hat{c}\hat{\gamma}} = T_{\hat{\alpha}\hat{b}}^i \hat{c} = 0 \quad (\text{dim } \frac{1}{2}) \quad (2.8b)$$

$$T_{\hat{a}\hat{b}}^{\hat{c}} = T_{\hat{a}\hat{\beta}(j} \hat{\beta}^{\hat{c}}_{k)} = 0 \quad (\text{dim } 1). \quad (2.8c)$$

The set of constraints (2.8a)–(2.8c) is obtained from that defining the minimal supergravity multiplet [3] by removing those constraints which correspond to the central-charge field strength.

With the constraints introduced, it can be shown that the torsion and the curvature tensors in (2.7) are expressed in terms of a small number of dimension-1 tensor superfields, S^{ij} , $X_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$ and $C_{\hat{a}}^{ij}$, and their covariant derivatives, with the symmetry properties:

$$S^{ij} = S^{ji}, \quad X_{\hat{a}\hat{b}} = -X_{\hat{b}\hat{a}}, \quad N_{\hat{a}\hat{b}} = -N_{\hat{b}\hat{a}}, \quad C_{\hat{a}}^{ij} = C_{\hat{a}}^{ji}. \quad (2.9)$$

Their reality properties are

$$\overline{S^{ij}} = S_{ij}, \quad \overline{X_{\hat{a}\hat{b}}} = X_{\hat{a}\hat{b}}, \quad \overline{N_{\hat{a}\hat{b}}} = N_{\hat{a}\hat{b}}, \quad \overline{C_{\hat{a}}^{ij}} = C_{\hat{a}ij}. \quad (2.10)$$

⁴The operation of (anti)symmetrization of n indices is defined to involve a factor $(n!)^{-1}$.

The covariant derivatives obey the (anti)commutation relations:

$$\begin{aligned} \{\mathcal{D}_{\hat{\alpha}}^i, \mathcal{D}_{\hat{\beta}}^j\} &= -2i \varepsilon^{ij} \mathcal{D}_{\hat{\alpha}\hat{\beta}} - i \varepsilon_{\hat{\alpha}\hat{\beta}} \varepsilon^{ij} X^{\hat{c}\hat{d}} M_{\hat{c}\hat{d}} + \frac{i}{4} \varepsilon^{ij} \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} (\Gamma_{\hat{a}})_{\hat{\alpha}\hat{\beta}} N_{\hat{b}\hat{c}} M_{\hat{d}\hat{e}} \\ &\quad - \frac{i}{2} \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} (\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}} C_{\hat{c}}^{ij} M_{\hat{d}\hat{e}} + 4i S^{ij} M_{\hat{\alpha}\hat{\beta}} + 3i \varepsilon_{\hat{\alpha}\hat{\beta}} \varepsilon^{ij} S^{kl} J_{kl} \\ &\quad - i \varepsilon^{ij} C_{\hat{\alpha}\hat{\beta}}^{kl} J_{kl} - 4i \left(X_{\hat{\alpha}\hat{\beta}} + N_{\hat{\alpha}\hat{\beta}} \right) J^{ij}, \end{aligned} \quad (2.11a)$$

$$\begin{aligned} [\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{\beta}}^j] &= \frac{1}{2} \left((\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} S^j_k - X_{\hat{a}\hat{b}} (\Gamma^{\hat{b}})_{\hat{\beta}}^{\hat{\gamma}} \delta_k^j - \frac{1}{4} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} N^{\hat{d}\hat{e}} (\Sigma^{\hat{b}\hat{c}})_{\hat{\beta}}^{\hat{\gamma}} \delta_k^j + (\Sigma_{\hat{a}}^{\hat{b}})_{\hat{\beta}}^{\hat{\gamma}} C_{\hat{b}}^{j k} \right) \mathcal{D}_{\hat{\gamma}}^k \\ &\quad - \frac{i}{2} \left((\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} T^{\hat{c}\hat{d}j}_{\hat{\gamma}} + 2(\Gamma^{\hat{c}})_{\hat{\beta}}^{\hat{\gamma}} T_{\hat{a}}^{\hat{d}j}_{\hat{\gamma}} \right) M_{\hat{c}\hat{d}} \\ &\quad + \left(3\Xi_{\hat{a}\hat{\beta}}^{(k \varepsilon^l)j} - \frac{1}{3} C_{\hat{a}\hat{\beta}}^{(k \varepsilon^l)j} - \frac{5}{4} (\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} \mathcal{F}_{\hat{\gamma}}^{(k \varepsilon^l)j} + \frac{1}{4} (\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} \mathcal{N}_{\hat{\gamma}}^{(k \varepsilon^l)j} \right. \\ &\quad \left. + \frac{1}{8} (\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} \mathcal{C}_{\hat{\gamma}}^{jkl} - \frac{11}{24} (\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} \mathcal{C}_{\hat{\gamma}}^{(k \varepsilon^l)j} \right) J_{kl}. \end{aligned} \quad (2.11b)$$

The dimension-1 components of the torsion, S^{ij} , $X_{\hat{a}\hat{b}}$, $N_{\hat{a}\hat{b}}$ and $C_{\hat{a}}^{ij}$, enjoy some additional differential constraints which follow from the Bianchi identities. To formulate them, it is useful to introduce the irreducible components of $\mathcal{D}_{\hat{\gamma}}^k X_{\hat{a}\hat{b}}$ and $\mathcal{D}_{\hat{\gamma}}^k C_{\hat{a}}^{ij}$ defined as follows:

$$\begin{aligned} \mathcal{D}_{\hat{\gamma}}^k X_{\hat{a}\hat{b}} &= W_{\hat{a}\hat{b}\hat{\gamma}}^k + 2(\Gamma_{[\hat{a}})_{\hat{\gamma}}^{\hat{\delta}} \Xi_{\hat{b}]\hat{\delta}} + (\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}}^{\hat{\delta}} \mathcal{F}_{\hat{\delta}}^k, \\ (\Gamma_{\hat{\alpha}})_{\hat{\alpha}}^{\hat{\beta}} \Xi_{\hat{a}\hat{\beta}}^i &= (\Gamma_{\hat{\alpha}})_{\hat{\alpha}}^{\hat{\beta}} W_{\hat{a}\hat{b}\hat{\beta}}^i = 0, \end{aligned} \quad (2.12a)$$

$$\begin{aligned} \mathcal{D}_{\hat{\gamma}}^k C_{\hat{a}}^{ij} &= C_{\hat{a}\hat{\gamma}}^{ijk} - \frac{2}{3} C_{\hat{a}\hat{\gamma}}^{(i \varepsilon^j)k} - \frac{1}{2} (\Gamma_{\hat{a}})_{\hat{\gamma}}^{\hat{\delta}} \mathcal{C}_{\hat{\delta}}^{ijk} + \frac{1}{3} (\Gamma_{\hat{a}})_{\hat{\gamma}}^{\hat{\delta}} \mathcal{C}_{\hat{\delta}}^{(i \varepsilon^j)k}, \\ C_{\hat{a}\hat{\gamma}}^{ijk} &= C_{\hat{a}\hat{\gamma}}^{(ijk)}, \quad \mathcal{C}_{\hat{\delta}}^{ijk} = \mathcal{C}_{\hat{\delta}}^{(ijk)}, \quad (\Gamma_{\hat{\alpha}})_{\hat{\alpha}}^{\hat{\beta}} C_{\hat{a}\hat{\beta}}^{ijk} = 0. \end{aligned} \quad (2.12b)$$

The dimension-3/2 Bianchi identities are:

$$\mathcal{D}_{\hat{\gamma}}^k N_{\hat{a}\hat{b}} = -W_{\hat{a}\hat{b}\hat{\gamma}}^k + 4(\Gamma_{[\hat{a}})_{\hat{\gamma}}^{\hat{\delta}} \Xi_{\hat{b}]\hat{\delta}} + (\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}}^{\hat{\delta}} \mathcal{N}_{\hat{\delta}}^k, \quad (2.13a)$$

$$C_{\hat{a}\hat{\gamma}}^{ijk} = 0, \quad (2.13b)$$

$$\mathcal{D}_{\hat{\gamma}}^k S^{ij} = -\frac{1}{4} C_{\hat{\gamma}}^{ijk} + \frac{5}{12} \mathcal{C}_{\hat{\gamma}}^{(i \varepsilon^j)k} + \frac{1}{2} \left(3\mathcal{F}_{\hat{\gamma}}^{(i} + \mathcal{N}_{\hat{\gamma}}^{(i} \right) \varepsilon^{j)k}. \quad (2.13c)$$

The dimension-3/2 torsion is

$$T_{\hat{a}\hat{b}\hat{\gamma}}^k = \frac{i}{2} \mathcal{D}_{\hat{\gamma}}^k X_{\hat{a}\hat{b}} - \frac{i}{6} (\Gamma_{[\hat{a}})_{\hat{\gamma}}^{\hat{\delta}} \mathcal{C}_{\hat{b}]\hat{\delta}}^k + \frac{i}{4} (\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}}^{\hat{\delta}} \mathcal{C}_{\hat{\delta}}^k. \quad (2.14)$$

The irreducible components of $\mathcal{D}_{\hat{\gamma}}^k N_{\hat{a}\hat{b}}$ are defined similarly to (2.12a). In accordance with eq. (2.13a), only one of them, $\mathcal{N}_{\hat{\delta}}^k$, is a new superfield, while the other two components occur in (2.12a). It is worth pointing out that eq. (2.13c) implies

$$\mathcal{D}_{\hat{\gamma}}^{(i} S^{j)k} = -\frac{1}{4} C_{\hat{\gamma}}^{ijk}. \quad (2.15)$$

The latter result will be important for our consideration below.

2.2 Super-Weyl transformations

A short calculation shows that the constraints (2.8a)–(2.8c) are invariant under super-Weyl transformations of the form:

$$\delta_\sigma \mathcal{D}_{\hat{\alpha}}^i = \sigma \mathcal{D}_{\hat{\alpha}}^i + 4(\mathcal{D}^{\hat{\gamma}i} \sigma) M_{\hat{\gamma}\hat{\alpha}} - 6(\mathcal{D}_{\hat{\alpha}k} \sigma) J^{ki}, \quad (2.16a)$$

$$\delta_\sigma \mathcal{D}_{\hat{a}} = 2\sigma \mathcal{D}_{\hat{a}} + i(\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}} (\mathcal{D}_{\hat{\gamma}}^k \sigma) \mathcal{D}_{\hat{\delta}k} - 2(\mathcal{D}^{\hat{b}} \sigma) M_{\hat{a}\hat{b}} + \frac{i}{4} (\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}} (\mathcal{D}_{\hat{\gamma}}^{(k} \mathcal{D}_{\hat{\delta}}^{l)} \sigma) J_{kl}, \quad (2.16b)$$

where the parameter $\sigma(z)$ is a real unconstrained superfield. The components of the torsion can be seen to transform as follows:

$$\delta_\sigma S^{ij} = 2\sigma S^{ij} + \frac{i}{2} \mathcal{D}^{\hat{\alpha}(i} \mathcal{D}_{\hat{\alpha}}^{j)} \sigma, \quad (2.17a)$$

$$\delta_\sigma C_{\hat{a}}^{ij} = 2\sigma C_{\hat{a}}^{ij} + i(\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}} \mathcal{D}_{\hat{\gamma}}^{(i} \mathcal{D}_{\hat{\delta}}^{j)} \sigma, \quad (2.17b)$$

$$\delta_\sigma X_{\hat{a}\hat{b}} = 2\sigma X_{\hat{a}\hat{b}} - \frac{i}{2} (\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}} \mathcal{D}_{\hat{\alpha}}^k \mathcal{D}_{\hat{\beta}k} \sigma, \quad (2.17c)$$

$$\delta_\sigma N_{\hat{a}\hat{b}} = 2\sigma N_{\hat{a}\hat{b}} - i(\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}} \mathcal{D}_{\hat{\alpha}}^k \mathcal{D}_{\hat{\beta}k} \sigma. \quad (2.17d)$$

It follows that

$$W_{\hat{a}\hat{b}} := X_{\hat{a}\hat{b}} - \frac{1}{2} N_{\hat{a}\hat{b}} \quad (2.18)$$

transforms homogeneously, and hence it can be identified with a superspace generalization of the Weyl tensor.

Let us analyze the supergravity multiplet introduced above. First, it consists of the fields that constitute the covariant derivatives (2.1) subject to the constraints (2.8a)–(2.8c). Second, it possesses the gauge freedom (2.5), (2.16a) and (2.16b). It proves to describe the Weyl multiplet [8, 9]. Indeed, one can choose a Wess-Zumino gauge and partially fix the super-Weyl gauge freedom in such a way that the remaining fields and the residual gauge transformations match those characteristic of the Weyl multiplet [8, 9]. In particular, it follows immediately from (2.17a), (2.17b) and (2.17c) that the θ -independent components of S^{ij} , $C_{\hat{a}}^{ij}$ and $X_{\hat{a}\hat{b}}$ can be gauged away by super-Weyl transformations. Such an approach is quite laborious. Fortunately, there is a more elegant way to prove the claim. It is sufficient to demonstrate that one ends up with the minimal supergravity multiplet [3] by coupling the above multiplet to an Abelian vector multiplet and then breaking the super-Weyl invariance. This will be demonstrated in the remainder of this section.

2.3 Coupling to a vector multiplet

Let us couple the Weyl multiplet to an Abelian vector multiplet. The covariant derivatives should be modified as follows:

$$\mathcal{D}_{\hat{A}} \longrightarrow \mathcal{D}_{\hat{A}} := \mathcal{D}_{\hat{A}} + V_{\hat{A}} Z, \quad (2.19)$$

with $V_{\hat{A}}(z)$ the gauge connection. We will interpret the generator Z to be a real central charge. It is also necessary to impose covariant constraints on some components of the field strength of the vector multiplet as in the flat case [4] (see also [25, 26]).

The covariant derivatives now satisfy the algebra

$$[\mathcal{D}_{\hat{A}}, \mathcal{D}_{\hat{B}}] = T_{\hat{A}\hat{B}}^{\hat{C}} \mathcal{D}_{\hat{C}} + \frac{1}{2} R_{\hat{A}\hat{B}}^{\hat{c}\hat{d}} M_{\hat{c}\hat{d}} + R_{\hat{A}\hat{B}}^{kl} J_{kl} + F_{\hat{A}\hat{B}} Z, \quad (2.20)$$

where the torsion and curvature are the same as before and the central charge field strengths are

$$F_{\hat{\alpha}\hat{\beta}}^{ij} = -2i\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}} W, \quad F_{\hat{a}\hat{b}}^j = (\Gamma_{\hat{a}})_{\hat{\beta}}^{\hat{\gamma}} \mathcal{D}_{\hat{\gamma}}^j W, \quad (2.21a)$$

$$F_{\hat{a}\hat{b}} = X_{\hat{a}\hat{b}} W + \frac{i}{4} (\Sigma_{\hat{a}\hat{b}})^{\hat{\gamma}\hat{\delta}} \mathcal{D}_{\hat{\gamma}}^k \mathcal{D}_{\hat{\delta}k} W. \quad (2.21b)$$

Here the field strength W is real, $\bar{W} = W$, and obeys the Bianchi identity

$$\mathcal{D}_{\hat{\alpha}}^{(i} \mathcal{D}_{\hat{\beta}}^{j)} W - \frac{1}{4} \varepsilon_{\hat{\alpha}\hat{\beta}} \mathcal{D}^{\hat{\gamma}(i} \mathcal{D}_{\hat{\gamma}}^{j)} W = \frac{i}{2} C_{\hat{\alpha}\hat{\beta}}^{ij} W. \quad (2.22)$$

The field strength W possesses the following super-Weyl transformation:

$$\delta_{\sigma} W = 2\sigma W. \quad (2.23)$$

It is a simple calculation to demonstrate that eq. (2.22) is invariant under the super-Weyl transformations.

2.4 The minimal multiplet

Suppose that the field strength of the vector multiplet is everywhere non-vanishing, $\langle W \rangle \neq 0$, that is the body of $W(z) \neq 0$ for any point $z \in \mathcal{M}^{5|8}$. Then, the super-Weyl symmetry can be used to choose the gauge

$$W = 1. \quad (2.24)$$

Now, eq. (2.22) reduces to

$$C_{\hat{a}}^{ij} = 0, \quad (2.25)$$

while eqs. (2.21a) and (2.21b) turn into

$$F_{\hat{\alpha}\hat{\beta}}^{ij} = -2i\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}}, \quad F_{\hat{a}\hat{b}}^j = 0, \quad F_{\hat{a}\hat{b}} = X_{\hat{a}\hat{b}}. \quad (2.26)$$

As a result, one ends up with the minimal supergravity multiplet [3]. This is analogous to the situation in 4D $\mathcal{N} = 2$ supergravity [27, 28].

3. Variant formulations for the Weyl multiplet

The fact that the super-Weyl gauge freedom allows one to gauge away $C_{\hat{a}}^{ij}$, eq. (2.25), is equivalent to the existence of an alternative formulation for the Weyl multiplet.

3.1 Reduced formulation

Let us start again from the superspace formulation for the Weyl multiplet we have developed in the previous section. We are in a position to choose the super-Weyl gauge (2.25). This is equivalent to the replacement of the dimension-1 constraints (2.8c) with

$$T_{\hat{a}\hat{b}}^{\hat{c}} = T_{\hat{a}\hat{\beta}(j}^{\hat{\beta}k)} = T_{\hat{a}(\hat{\beta}\hat{\gamma}}^{(j k)} = 0 . \quad (3.1)$$

Then, eq. (2.15) turns into

$$\mathcal{D}_{\hat{\alpha}}^{(i} S^{jk)} = 0 . \quad (3.2)$$

In accordance with (2.17b), the residual super-Weyl transformations are generated by a parameter obeying the constraint

$$\mathcal{D}_{\hat{\alpha}}^{(i} \mathcal{D}_{\hat{\beta}}^{j)} \sigma - \frac{1}{4} \varepsilon_{\hat{\alpha}\hat{\beta}} \mathcal{D}^{\hat{\gamma}(i} \mathcal{D}_{\hat{\gamma}}^{j)} \sigma = 0 . \quad (3.3)$$

Clearly, this (partially gauged-fixed) superspace setting still describes the Weyl multiplet. However, the present formulation is technically much simpler to deal with than the one developed in the previous section. In what follows, we will only use the formulation for the Weyl multiplet which is given in the present section. It will be referred to as the *reduced formulation*.

3.2 Coupling to a vector multiplet

If an Abelian vector multiplet is coupled to the Weyl multiplet, the corresponding field strength W obeys the Bianchi identity

$$\mathcal{D}_{\hat{\alpha}}^{(i} \mathcal{D}_{\hat{\beta}}^{j)} W - \frac{1}{4} \varepsilon_{\hat{\alpha}\hat{\beta}} \mathcal{D}^{\hat{\gamma}(i} \mathcal{D}_{\hat{\gamma}}^{j)} W = 0 \quad (3.4)$$

which is obtained from (2.22) by setting $C_{\hat{\alpha}\hat{\beta}}^{ij} = 0$. Comparing (3.3) with (3.4), we see that W and the super-Weyl parameter are constrained superfields of the same type.

Let us consider the composite superfield

$$G^{ij} := i \mathcal{D}^{\hat{\alpha}(i} W \mathcal{D}_{\hat{\alpha}}^{j)} W + \frac{i}{2} W \mathcal{D}^{ij} W - 2 S^{ij} W^2, \quad \mathcal{D}^{ij} := \mathcal{D}^{\hat{\alpha}(i} \mathcal{D}_{\hat{\alpha}}^{j)}, \quad (3.5)$$

which is a curved superspace extension of the composite $O(2)$ multiplet introduced in [26] and later in [29] for the flat and Anti-de Sitter cases, respectively. Its crucial property is

$$\mathcal{D}_{\hat{\alpha}}^{(i} G^{jk)} = 0 . \quad (3.6)$$

The super-Weyl transformation of G^{ij} can be shown, with the use of eqs. (2.16a) and (2.23), to be

$$\delta_{\sigma} G^{ij} = 6\sigma G^{ij} . \quad (3.7)$$

If the field strength W is everywhere non-vanishing, $\langle W \rangle \neq 0$, the super-Weyl gauge freedom can be used to choose the gauge (2.24). The result is again the minimal supergravity multiplet [3].

3.3 The dilaton Weyl multiplet

In superspace, the dilaton Weyl multiplet [8, 9] can be realized as the standard Weyl multiplet coupled to an Abelian vector multiplet such that its field strength W is everywhere non-vanishing, $\langle W \rangle \neq 0$, and enjoys the equation

$$G^{ij} = i\mathcal{D}^{\hat{\alpha}(i}W\mathcal{D}_{\hat{\alpha}}^{j)}W + \frac{i}{2}W\mathcal{D}^{ij}W - 2S^{ij}W^2 = 0. \quad (3.8)$$

The latter is equivalent to

$$S^{ij} = \frac{i}{2W^2} \left\{ \mathcal{D}^{\hat{\alpha}(i}W\mathcal{D}_{\hat{\alpha}}^{j)}W + \frac{1}{2}W\mathcal{D}^{ij}W \right\}. \quad (3.9)$$

Similarly to the rigid supersymmetric case [26], eq. (3.8) originates as the equation of motion in a Chern-Simons model for the vector multiplet.

It is not difficult to generalize the construction given. Suppose we have a system of $n+1$ Abelian vector multiplets, and let $W_a(z)$ be the corresponding field strengths, where $a = 0, 1, \dots, n$. Instead of the single composite object (3.5), we now have $(n+1)(n+2)/2$ such superfields defined as (compare with [21, 29])

$$G_{ab}^{ij} := i\mathcal{D}^{\hat{\gamma}(i}W_a\mathcal{D}_{\hat{\gamma}}^{j)}W_b + \frac{i}{2}W_{(a}\mathcal{D}^{ij}W_{b)} - 2S^{ij}W_aW_b, \quad \mathcal{D}_{\hat{\gamma}}^{(i}G_{ab}^{jk)} = 0. \quad (3.10)$$

Assume also that the field strength W_0 is everywhere non-vanishing, $\langle W_0 \rangle \neq 0$. Now, we can generalize eq. (3.8) as follows:

$$M^{ab}G_{ab}^{ij}(z) = 0, \quad M^{ab} = M^{ba}, \quad (3.11)$$

where M^{ab} is a constant nonsingular real matrix normalized as $M^{00} = 1$.

4. Projective supermultiplets

So far, we have provided the superspace realization for the main kinematic constructions of the superconformal tensor calculus [8, 9]. In the remainder of this paper, we present new results that have not appeared in the component approaches of [8, 9]. We first introduce a large family of new off-shell (matter) supermultiplets coupled to the Weyl multiplet of 5D $\mathcal{N} = 1$ conformal supergravity. They can be viewed to be a curved superspace generalization of the known off-shell multiplets in 4D $\mathcal{N} = 2$ flat projective superspace [18–20] or, more precisely, of the rigid 5D $\mathcal{N} = 1$ superconformal multiplets [21]. Their off-shell structure is almost identical to that of the supermultiplets coupled to the minimal supergravity multiplet, which we have proposed in [1]. Therefore, below we will closely follow [1] and specifically emphasize those features that are characteristic of conformal supergravity.

In addition to the superspace coordinates $z^{\hat{M}} = (x^{\hat{m}}, \theta_i^{\hat{t}})$, it is useful to introduce isotwistor variables $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$ defined to be inert with respect to the local group $SU(2)$ [1]. The operators $\mathcal{D}_{\hat{\alpha}}^+ := u_i^+ \mathcal{D}_{\hat{\alpha}}^i$ obey the following algebra:

$$\{\mathcal{D}_{\hat{\alpha}}^+, \mathcal{D}_{\hat{\beta}}^+\} = -4i \left(X_{\hat{\alpha}\hat{\beta}} + N_{\hat{\alpha}\hat{\beta}} \right) J^{++} + 4i S^{++} M_{\hat{\alpha}\hat{\beta}}, \quad (4.1)$$

where $J^{++} := u_i^+ u_j^+ J^{ij}$ and $S^{++} := u_i^+ u_j^+ S^{ij}$. Eq. (4.1) follows from (2.11a). It is tempting to consider constrained superfields $Q(z, u^+)$ obeying the constraint $\mathcal{D}_{\hat{\alpha}}^+ Q = 0$. For the latter to be consistent, $Q(z, u^+)$ must be scalar with respect to the Lorentz group, $M_{\hat{\alpha}\hat{\beta}} Q = 0$, and also possess special properties with respect to the group $SU(2)$, that is, $J^{++} Q = 0$. Let us define such supermultiplets.

A projective supermultiplet of weight n , $Q^{(n)}(z, u^+)$, is a scalar superfield that lives on $\mathcal{M}^{5|8}$, is holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

(i) it obeys the covariant analyticity constraint⁵

$$\mathcal{D}_{\hat{\alpha}}^+ Q^{(n)} = 0 ; \tag{4.2}$$

(ii) it is a homogeneous function of u^+ of degree n , that is,

$$Q^{(n)}(z, c u^+) = c^n Q^{(n)}(z, u^+), \quad c \in \mathbb{C}^* ; \tag{4.3}$$

(iii) infinitesimal gauge transformations (2.5) act on $Q^{(n)}$ as follows:

$$\begin{aligned} \delta_K Q^{(n)} &= \left(K^{\hat{C}} \mathcal{D}_{\hat{C}} + K^{ij} J_{ij} \right) Q^{(n)}, \\ K^{ij} J_{ij} Q^{(n)} &= -\frac{1}{(u^+ u^-)} \left(K^{++} D^{--} - n K^{+-} \right) Q^{(n)}, \quad K^{\pm\pm} = K^{ij} u_i^{\pm} u_j^{\pm}, \end{aligned} \tag{4.4}$$

where

$$D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}, \quad D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}. \tag{4.5}$$

The transformation law (4.4) involves an additional isotwistor, u_i^- , which is subject to the only condition $(u^+ u^-) = u^{+i} u_i^- \neq 0$, and is otherwise completely arbitrary. By construction, $Q^{(n)}$ is independent of u^- , i.e. $\partial Q^{(n)} / \partial u^{-i} = 0$, and hence $D^{++} Q^{(n)} = 0$. One can see that $\delta Q^{(n)}$ is also independent of the isotwistor u^- , $\partial(\delta Q^{(n)}) / \partial u^{-i} = 0$, due to (4.3). It follows from (4.4)

$$J^{++} Q^{(n)} = 0, \quad J^{++} \propto D^{++}, \tag{4.6}$$

and hence the covariant analyticity constraint (4.2) is indeed consistent.

In conformal supergravity, the important issue is how the projective multiplets may consistently vary under the super-Weyl transformations. Suppose we are given a weight- n projective superfield $Q^{(n)}$ that transforms homogeneously, $\delta_{\sigma} Q^{(n)} \propto \sigma Q^{(n)}$. Then, its transformation law turns out to be uniquely fixed by the constraint (4.2).

$$\delta_{\sigma} Q^{(n)} = 3n \sigma Q^{(n)}. \tag{4.7}$$

The super-Weyl weight, $3n$, matches the superconformal weight of a rigid superconformal projective multiplet [21] to which $Q^{(n)}$ reduces in the flat superspace limit. Without the

⁵In the case of rigid $\mathcal{N} = 2$ supersymmetry in four dimensions, similar constraints were first introduced by Rosly [30], and later by the harmonic [11] and projective [18, 19] superspace practitioners.

assumption of homogeneity, it is easy to construct examples of projective multiplets which do not respect (4.7). For instance, It follows from (3.2) that S^{++} is a projective superfield of weight two,

$$\mathcal{D}_{\hat{\alpha}}^+ S^{++} = 0 . \tag{4.8}$$

In accordance with (2.17a), its super-Weyl transformation is

$$\delta_{\sigma} S^{++} = 2\sigma S^{++} + \frac{i}{2} (\mathcal{D}^+)^2 \sigma , \quad (\mathcal{D}^+)^2 := \mathcal{D}^{+\hat{\alpha}} \mathcal{D}_{\hat{\alpha}}^+ . \tag{4.9}$$

Given a projective multiplet $Q^{(n)}$, its complex conjugate is not covariantly analytic. However, similarly to the flat four-dimensional case [30, 11, 18] , one can introduce a generalized, analyticity-preserving conjugation, $Q^{(n)} \rightarrow \tilde{Q}^{(n)}$, defined as

$$\tilde{Q}^{(n)}(u^+) \equiv \bar{Q}^{(n)}(\bar{u}^+ \rightarrow \tilde{u}^+) , \quad \tilde{u}^+ = i\sigma_2 u^+ , \tag{4.10}$$

with $\bar{Q}^{(n)}(\bar{u}^+)$ the complex conjugate of $Q^{(n)}$. Its fundamental property is

$$\widetilde{\mathcal{D}_{\hat{\alpha}}^+ Q^{(n)}} = (-1)^{\epsilon(Q^{(n)})} \mathcal{D}^{+\hat{\alpha}} \tilde{Q}^{(n)} . \tag{4.11}$$

One can see that $\widetilde{\tilde{Q}^{(n)}} = (-1)^n Q^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. In what follows, $\tilde{Q}^{(n)}$ will be called the smile-conjugate of $Q^{(n)}$.

Important examples of projective supermultiplets are given in [1], and we refer the reader to that paper for more details.

Let W be the field strength of an Abelian vector multiplet. We can then introduce

$$G^{++} := G^{ij} u_i^+ u_j^+ = i\mathcal{D}^{+\hat{\alpha}} W \mathcal{D}_{\hat{\alpha}}^+ W + \frac{i}{2} W (\mathcal{D}^+)^2 W - 2S^{++} W^2 , \tag{4.12}$$

with G^{ij} defined in (3.5). It follows from (3.6) that G^{++} is a projective superfield of weight two,

$$\mathcal{D}_{\hat{\alpha}}^+ G^{++} = 0 . \tag{4.13}$$

In accordance with (3.7), the super-Weyl transformation of G^{++} conforms with (4.7),

$$\delta_{\sigma} G^{++} = 6\sigma G^{++} . \tag{4.14}$$

Consider a given supergravity background. The superconformal group of this space is defined to be generated by those combined infinitesimal transformations (2.5), (2.16a) and (2.16b) which do not change the covariant derivatives,

$$\delta_K \mathcal{D}_{\hat{A}} + \delta_{\sigma} \mathcal{D}_{\hat{A}} = 0 . \tag{4.15}$$

This definition is analogous to that often used in 4D $\mathcal{N} = 1$ supergravity [31]. In the case of 5D $\mathcal{N} = 1$ flat superspace, it is equivalent to the definition of the superconformal Killing vectors [21]. In this case, the transformation laws of the projective multiplets reduce to those describing the rigid superconformal projective multiplets [21].

In defining the projective supermultiplets, we have used the reduced formulation for the Weyl multiplet. It is not difficult to see that this definition remains valid if the superspace

geometry is realized in terms of the formulation for the Weyl multiplet presented in section 2. Indeed, from (2.11a) one deduces the anti-commutation relation:

$$\{\mathcal{D}_{\hat{\alpha}}^+, \mathcal{D}_{\hat{\beta}}^+\} = -4i \left(X_{\hat{\alpha}\hat{\beta}} + N_{\hat{\alpha}\hat{\beta}} \right) J^{++} + 4i S^{++} M_{\hat{\alpha}\hat{\beta}} - \frac{i}{2} \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} (\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}} C_{\hat{c}}^{++} M_{\hat{d}\hat{e}}, \quad (4.16)$$

where $C_{\hat{a}}^{++} := C_{\hat{a}}^{ij} u_i^+ u_j^+$. It implies that the constraint (4.2) is consistent under the same conditions on $Q^{(n)}(z, u^+)$ which we have specified above.

5. Action principle

Let \mathcal{L}^{++} be a real projective multiplet of weight two, and W the field strength of a vector multiplet such that $\langle W \rangle \neq 0$. We assume that \mathcal{L}^{++} possesses the following super-Weyl transformation:

$$\delta_{\sigma} \mathcal{L}^{++} = 6\sigma \mathcal{L}^{++} \quad (5.1)$$

which complies with (4.7). Associated with \mathcal{L}^{++} is the following functional

$$S(\mathcal{L}^{++}) = \frac{2}{3\pi} \oint (u^+ du^+) \int d^5x d^8\theta E \frac{\mathcal{L}^{++} W^4}{(G^{++})^2}, \quad E^{-1} = \text{Ber}(E_{\hat{A}}^{\hat{M}}). \quad (5.2)$$

This functional is invariant under arbitrary re-scalings $u_i^+(t) \rightarrow c(t) u_i^+(t)$, $\forall c(t) \in \mathbb{C} \setminus \{0\}$, where t denotes the evolution parameter along the integration contour. We are going to demonstrate that $S(\mathcal{L}^{++})$ does not change under the supergravity gauge transformations and is super-Weyl invariant. Therefore, eq. (5.2) constitutes a locally supersymmetric and super-Weyl invariant action principle.

To prove the invariance of $S(\mathcal{L}^{++})$ under infinitesimal supergravity gauge transformations (2.5) and (4.4), we first point out that

$$Q^{(-2)} := \frac{\mathcal{L}^{++}}{(G^{++})^2} \quad (5.3)$$

is a projective multiplet of weight -2 , because both \mathcal{L}^{++} and G^{++} are projective multiplet of weight $+2$. Since W is SU(2)-scalar and u -independent, from eq. (4.4) we can deduce (see also [2])

$$K^{ij} J_{ij} \left(Q^{(-2)} W^4 \right) = -\frac{1}{(u^+ u^-)} D^{--} \left(K^{++} Q^{(-2)} W^4 \right). \quad (5.4)$$

Next, since $K^{++} Q^{(-2)}$ has weight zero, it is easy to see

$$(u^+ du^+) K^{ij} J_{ij} \left(Q^{(-2)} W^4 \right) = -dt \frac{d}{dt} \left(Q^{(-2)} W^4 \right), \quad (5.5)$$

with t the evolution parameter along the integration contour in (5.2). Since the integration contour is closed, the SU(2)-part of the transformation (4.4) does not contribute to the variation of the action (5.2). To complete the proof of local supersymmetry invariance, it remains to take into the account the fact that $\mathcal{L}^{++}/(G^{++})^2$ and W are Lorentz scalars.

To prove the invariance of $S(\mathcal{L}^{++})$ under the infinitesimal super-Weyl transformations, we first note the transformation law of E :

$$\delta_\sigma E = -2\sigma E . \tag{5.6}$$

Now, it only remains to take into account the transformation laws (2.23), (4.14) and (5.1).

Let us introduce the following fourth-order operator⁶ (see also [2]):

$$\Delta^{(+4)} = (\mathcal{D}^+)^4 - \frac{5}{12}i S^{++} (\mathcal{D}^+)^2 + 3(S^{++})^2 , \tag{5.7}$$

where

$$(\mathcal{D}^+)^4 := -\frac{1}{96}\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\mathcal{D}_{\hat{\alpha}}^+\mathcal{D}_{\hat{\beta}}^+\mathcal{D}_{\hat{\gamma}}^+\mathcal{D}_{\hat{\delta}}^+ . \tag{5.8}$$

Its crucial property is that the superfield $Q^{(n)}$ defined by

$$Q^{(n)}(z, u^+) := \Delta^{(+4)}U^{(n-4)}(z, u^+) , \tag{5.9}$$

is a weight- n projective multiplet,

$$\mathcal{D}_{\hat{\alpha}}^+Q^{(n)} = 0 , \tag{5.10}$$

for any *unconstrained* scalar superfield $U^{(n-4)}(z, u^+)$ that lives on $\mathcal{M}^{5|8}$, is holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

(i) it is a homogeneous function of u^+ of degree $n - 4$, that is,

$$U^{(n-4)}(z, c u^+) = c^{n-4} U^{(n-4)}(z, u^+) , \quad c \in \mathbb{C}^* ; \tag{5.11}$$

(iii) infinitesimal gauge transformations (2.5) act on $U^{(n-4)}$ as follows:

$$\begin{aligned} \delta_K U^{(n-4)} &= \left(K^{\hat{C}} \mathcal{D}_{\hat{C}} + K^{ij} J_{ij} \right) U^{(n-4)} , \\ K^{ij} J_{ij} U^{(n-4)} &= -\frac{1}{(u^+ u^-)} \left(K^{++} D^{--} - (n-4) K^{+-} \right) U^{(n-4)} . \end{aligned} \tag{5.12}$$

We will call $U^{(n-4)}(z, u^+)$ a *projective prepotential* for $Q^{(n)}$. It can be checked that $U^{(n-4)}$ should possess the super-Weyl transformation

$$\delta_\sigma U^{(n-4)} = (3n - 4)\sigma U^{(n-4)} \tag{5.13}$$

in order for $Q^{(n)} = \Delta^{(+4)}U^{(n-4)}$ to transform as in (4.7).

The important result is

$$\Delta^{(+4)}W^4 = \frac{3}{4}(G^{++})^2 . \tag{5.14}$$

⁶This operator was considered for the first time in [29] in the case of 5D $\mathcal{N} = 1$ anti-de Sitter supersymmetry.

This relation can be proved by using the identity

$$\mathcal{D}_{\hat{\alpha}}^+ \mathcal{D}_{\hat{\beta}}^+ \mathcal{D}_{\hat{\gamma}}^+ W = -2i\varepsilon_{\hat{\beta}\hat{\gamma}} S^{++} \mathcal{D}_{\hat{\alpha}}^+ W \quad (5.15)$$

which follows from the Bianchi identity (3.4) with the aid of (4.1).

Let $U^{(-2)}$ be a projective prepotential for the Lagrangian \mathcal{L}^{++} in (5.2)

$$\mathcal{L}^{++} = \Delta^{(+4)} U^{(-2)} . \quad (5.16)$$

Using the rule for integration by parts

$$\int d^5x d^8\theta E \mathcal{D}_{\hat{A}} \Phi^{\hat{A}} = 0, \quad (5.17)$$

for an arbitrary superfield $\Phi^{\hat{A}} = (\Phi^{\hat{a}}, \Phi_i^{\hat{a}})$, we obtain

$$\frac{2}{3\pi} \oint (u^+ du^+) \int d^5x d^8\theta E \frac{\mathcal{L}^{++} W^4}{(G^{++})^2} = \frac{1}{2\pi} \oint (u^+ du^+) \int d^5x d^8\theta E U^{(-2)}, \quad (5.18)$$

where we have used (5.14). This crucial relation tells us that the supersymmetric action, eq. (5.2), is independent of the concrete choice of a vector multiplet with $\langle W \rangle \neq 0$, provided \mathcal{L}^{++} is independent of this vector multiplet.

It is worth pointing out that the super-Weyl invariance of the right-hand side in (5.18) also follows from (5.13).

Since the action (5.2) is super-Weyl invariant, one can choose the super-Weyl gauge (2.24). Then, due to the explicit form of G^{++} , eq. (4.12), the action reduces to the functional

$$S(\mathcal{L}^{++}) = \frac{1}{6\pi} \oint (u^+ du^+) \int d^5x d^8\theta E \frac{\mathcal{L}^{++}}{(S^{++})^2} \quad (5.19)$$

proposed in [2]. As demonstrated in [2], this functional is a natural extension of the action principle in flat projective superspace [18, 32].

It is useful to give several examples of supergravity-matters systems. Let $\mathbb{V}(z, u^+)$ denote the tropical prepotential for the central charge vector multiplet appearing in the action (5.2) (see [1] for more detail). It is a real weight-zero projective multiplet possessing the gauge invariance

$$\delta\mathbb{V} = \lambda + \tilde{\lambda}, \quad (5.20)$$

with λ a weight-zero arctic multiplet (see [1] for the definition of arctic multiplets). A hypermultiplet can be described by an arctic weight-one multiplet $\Upsilon^+(z, u^+)$ and its smile-conjugate $\tilde{\Upsilon}^+$. Consider a gauge invariant Lagrangian of the form (with the gauge transformation of Υ^+ being $\delta\Upsilon^+ = -\xi^{-1}\lambda$)

$$\mathcal{L}^{++} = \frac{1}{k^2} \mathbb{V} G^{++} - \tilde{\Upsilon}^+ e^{\xi\mathbb{V}} \Upsilon^+, \quad (5.21)$$

with κ Newton's constant, and ξ a cosmological constant. It describes Poincaré supergravity if $\xi = 0$, and pure gauge supergravity with $\xi \neq 0$.

A system of arctic weight-one multiplets $\Upsilon^+(z, u^+)$ and their smile-conjugates $\tilde{\Upsilon}^+$ can be described by the Lagrangian

$$\mathcal{L}^{++} = i K(\Upsilon^+, \tilde{\Upsilon}^+), \quad (5.22)$$

with $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ a real analytic function of n complex variables Φ^I , where $I = 1, \dots, n$. For \mathcal{L}^{++} to be a weight-two real projective superfield, it is sufficient to require

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}). \quad (5.23)$$

This is a curved superspace generalization of the general model for superconformal polar multiplets [21, 29, 22].

As a generalization of the model given in [1], a system of interacting arctic weight-zero multiplets Υ and their smile-conjugates $\tilde{\Upsilon}$ can be described by the Lagrangian

$$\mathcal{L}^{++} = G^{++} \mathbf{K}(\Upsilon, \tilde{\Upsilon}), \quad (5.24)$$

with $\mathbf{K}(\Phi^I, \bar{\Phi}^{\bar{J}})$ a real function which is not required to obey any homogeneity condition. The action is invariant under Kähler transformations of the form

$$\mathbf{K}(\Upsilon, \tilde{\Upsilon}) \rightarrow \mathbf{K}(\Upsilon, \tilde{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\tilde{\Upsilon}), \quad (5.25)$$

with $\Lambda(\Phi^I)$ a holomorphic function.

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